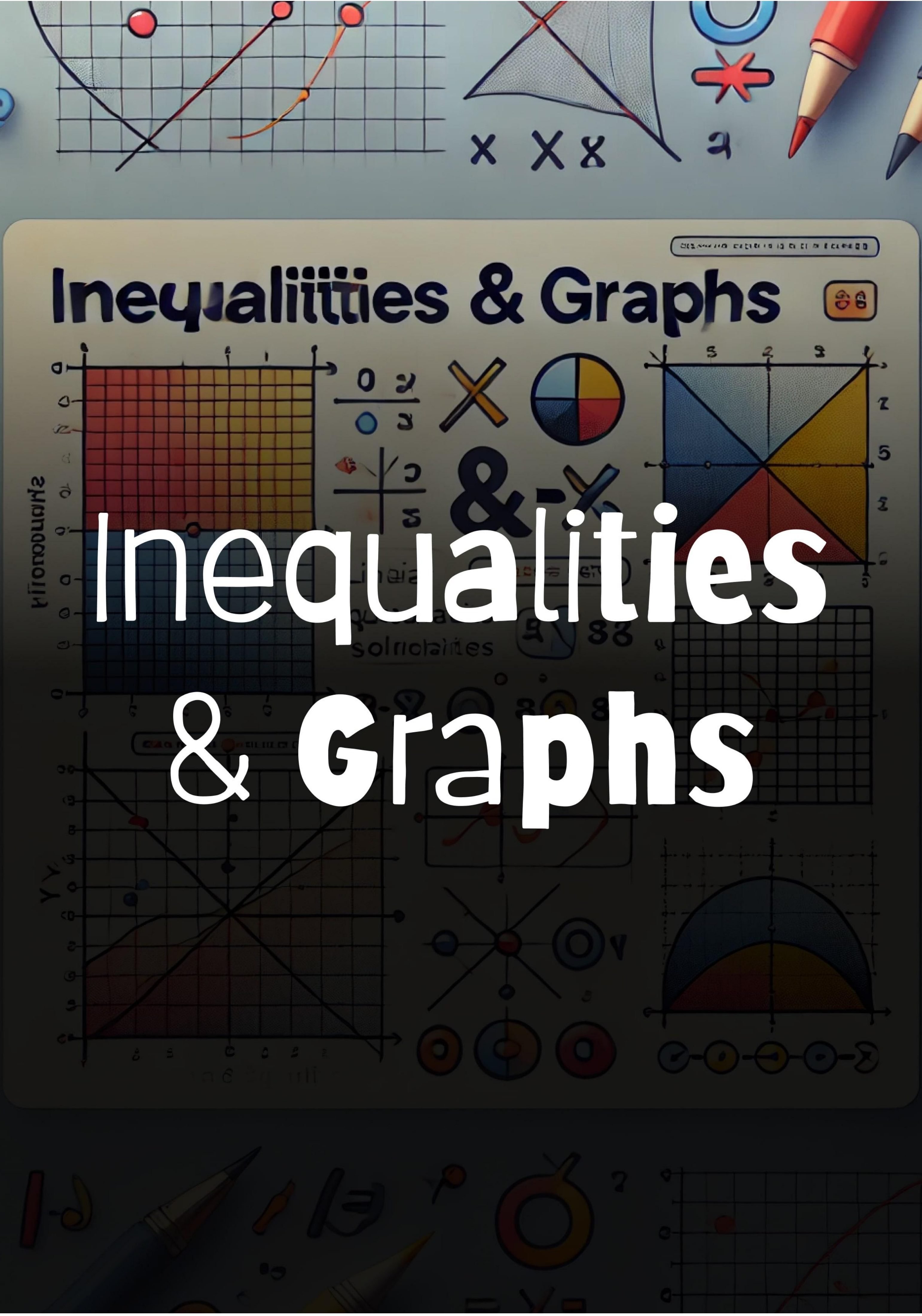


Inequalities & Graphs

INEQUALITIES & GRAPHS

Ques: $(3-x)(x-2)(x-4)(5-x) \le 0$, find the solution set of above inequality.

Sol: $(x-3)(x-2)(x-4)(x-5) \le 0$

$$\boxed{x \in [2,3] \cup [4,5]}$$

Rational Inequality

Ques: $(x^2+5x+6)(x^2+5x+4) \le 24$, find solution:

Sol: $(x^2+3x+2x+6)(x^2+4x+x+4) \le 24$

$\Rightarrow (x+3)(x+2)(x+4)(x+1) \le 24$

$\Rightarrow (x+6)(x+4) \le 24$

$\Rightarrow x^2+10x+24-24 \le 0$

$\Rightarrow x^2+10x \le 0$

$\Rightarrow (x^2+5x)^2+10(x^2+5x) \le 0$

$\Rightarrow x(x+10) \le 0$

$+ \quad - \quad +$; $-10 \quad 0$; $\in [-10, 0]$

$\Rightarrow -10 \le x^2+5x \le 0$

$\Rightarrow x^2+5x \ge -10$

$\Rightarrow x^2+5x+10 \ge 0$

$x^2+5x \le 0$

$-5 \quad 0$

$$\boxed{x \in [-5, 0]}$$

TYPE I

In order to solve the rational inequality, we use following steps:

STEP-1 Factor the expression into linear form and find the roots of the expression.

STEP-2 Check the coefficient of x in each term of the expression. If all are +ve, then go to step III directly, otherwise count the no. of −ve coefficients.

If the count is even → Make all the coefficient of x +ve without changing sign of inequality.

If the count is odd → Then make all the coefficient of x +ve and also change the sign of inequality

eg. 1. $(x-1)(x-2)(x-3)(x-4) \le 0$ ⇒ Move to step III

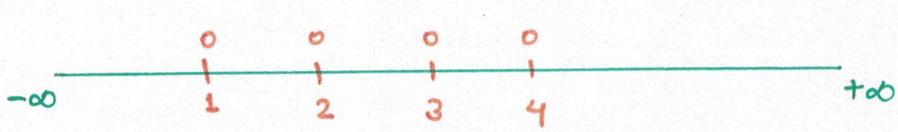
eg. 2. $(x-1)(2-x)(3-x)(4-x) \le 0$

$\Rightarrow (x-1)(x-2)(x-3)(x-4) \ge 0$

eg. 3. $(x-1)(x-2)(3-x)(4-x) \le 0$

$\Rightarrow (x-1)(x-2)(x-3)(x-4) \le 0$

STEP-3 Draw the real number line and plot the roots in increasing order.



STEP-4 Write + in the right most part of the no. line and change the sign after every root.

STEP-5 According to the inequality, find the solution set.

eg. $(x-1)(x-2)(x-3)(x-4) \geq 0$

$x \in (-\infty, 1] \cup [2,3] \cup [4, \infty)$

QUESTION

$(x^3-6x^2+11x-6)(x^3+6x^2+11x+6)(x^2+9x+20) \leq 0$

Solution:

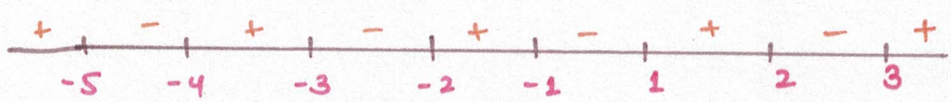
$[x^3+11x-(6x^2+6)]\,[x^3+6x^2+11x+6]\,(x+4)(x+5)<0$

Multiply & Divide $(x-1)$

$(x+1)$

$(x-1)(x^2-5x+6)(x+1)(x^2+5x+6)(x+4)(x+5)<0$

$(x-1)(x+1)(x+4)(x+5)(x-2)(x-3)(x+2)(x+3)<0$



$$\boxed{x \in (-5,-4) \cup (-3,-2) \cup (-1,1) \cup (2,3)}$$

POINTS TO REMEMBER

- RHS must be zero
- Cross multiplication and cancellation of common factors is prohibited
- If the sign of inequality is < or >. then, the solution set will have open bracket always. But if the sign of inequality is ≤ or ≥ , then the solution set may have closed brackets

101

TYPE-2

If the expression contains some quadratic terms which cannot be factorised. i.e, the discriminant $D<0$, then use the following steps to obtain solution set:

STEP-1 Find the roots of given expression after factorising each term except a quadratic equation whose $D<0$

eg. $(x^2-5x+6)(x^2-x+1)(x^2-4x)(x^2+1)<0$

$$\Rightarrow \underset{\downarrow\, 2}{(x-2)}\ \underset{\downarrow\, 3}{(x-3)}\ \underset{\downarrow\, 0}{(x)}\ \underset{\downarrow\, 4}{(x-4)}\ \underbrace{(x^2-x+1)\ (x^2+1)}_{D<0} < 0$$

STEP-2 If in step-I, we get some roots, then remove those quadratic expressions whose $D<0$ and $a<0$, then first make a positive by taking -ve sign common and then remove it.

If we are not getting any roots in step I, then use your BRAIN

Example (i) $\underset{\downarrow\, 2}{(x-2)}\ \underset{\downarrow\, 3}{(x-3)}\ \underset{\downarrow\, 0}{(x)}\ \underset{\downarrow\, 4}{(x-4)} < 0$

(ii) $(x-1)(x-2)(4-x)(-x^2-1)<0$

$(x-1)(x-2)(x-4)(x^2+1)<0$

$(x-1)(x-2)(x-4)<0$

(iii) $(x^2-x+1)(-x^2-1)<0$

$(x^2-x+1)(x^2+1)>0$

STEP-3 Check the coefficient of x in each term of the expression. If all coefficient are +ve, then go to step (iv), otherwise count the no. of -ve coefficient. If the count is even, then make them +ve without changing the sign of inequality and if the count is odd, then make them +ve by changing the sign of inequality.

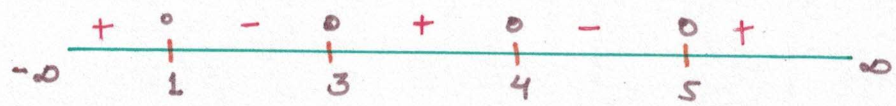
Example: (i) $(x-1)(x-3)(4-x)(5-x)(x^2+1) \le 0$

$$(x-1)(x-3)(x-4)(x-5) \le 0$$

(ii) $(x-1)(x-3)(4-x)(5-x)(-x^2-1) \le 0$

$$(x-1)(x-3)(x-4)(x-5) \ge 0$$

STEPS-4 Draw the real no. line and plot the critical points in increasing order.



STEP-5 Write + on the rightmost part of the no. line and change the sign after every critical point.

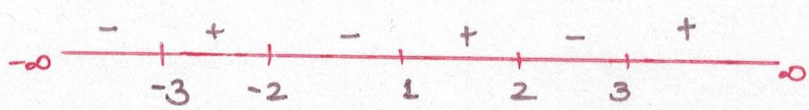
STEP-6 Write the answer as per the inequality

Ques: $(x^3-1)(x^4-16)(9-x^2) \le 0$

Sol: $(x-1)(x^2+x+1)(x^2-4)(x^2+4)(3-x)(3+x) \le 0$

$$(x-1)(x^2+x+1)(x+2)(x-2)(x-3)(x+3) \ge 0$$

$(x-1) \downarrow 1$, $(x+2) \downarrow -2$, $(x-2) \downarrow +2$, $(x-3) \downarrow +3$, $(x+3) \downarrow -3$

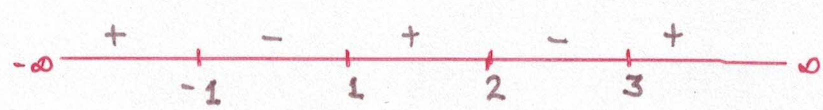


$$\boxed{x \in [-3,-2] \cup [1,2] \cup [3,\infty)}$$

Ques: $(x^2-5x+6)(x^3-1)(x^3+1) > 0$

Sol: $(x-2)(x-3)(x-1)(x^2+x+1)(x+1)(x^2+1-x) > 0$

$(x-2) \downarrow 2$, $(x-3) \downarrow 3$, $(x-1) \downarrow 1$, $(x+1) \downarrow -1$



$x \in (-\infty, -1) \cup (1, 2) \cup (3, \infty)$

HOW TO DRAW THE GRAPH

In order to obtain the graph of $f(x)$, we use 3 step process:

Step 1 Find the domain, range of the given function and some benchmarks on coordinates axis from where the graph of $f(x)$ must pass.

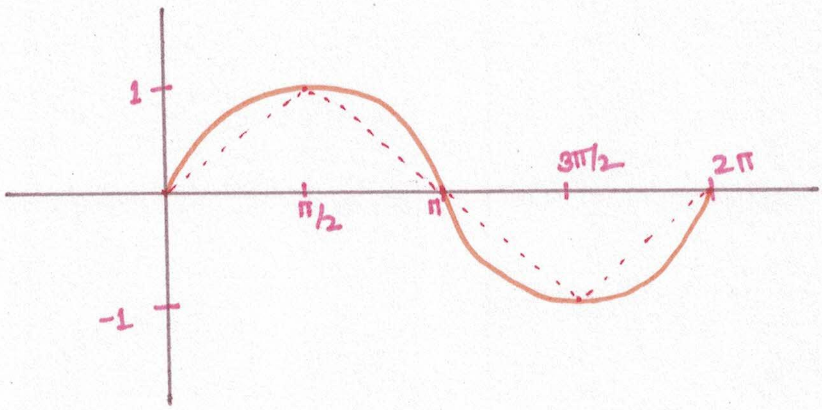
Step 2 In this step, we will try to find the path of the graph for this differentiate the f^n within the dom-ain and check the sign of output.

1. If $\frac{dy}{dx} > 0$, in the entire domain, then graph of $f(x)$ will increase
2. If $\frac{dy}{dx} < 0$, the graph of $f(x)$ will decrease
3. If $\frac{dy}{dx} = 0$, the graph of $f(x)$ will be parallel to x-axis

Step 3 In this step, we decide the shape of the graph. For this, double differentiate $f(x)$ within the domain and check the sign of output.

1. If $\frac{d^2y}{d^2x} > 0$, $f(x)$ will be concave upwards
2. If $\frac{d^2y}{d^2x} < 0$, then concave downwards
3. If $\frac{d^2y}{d^2x} = 0$, then straight line

Question: $y = \sin x \quad x \in [0, 2\pi]$

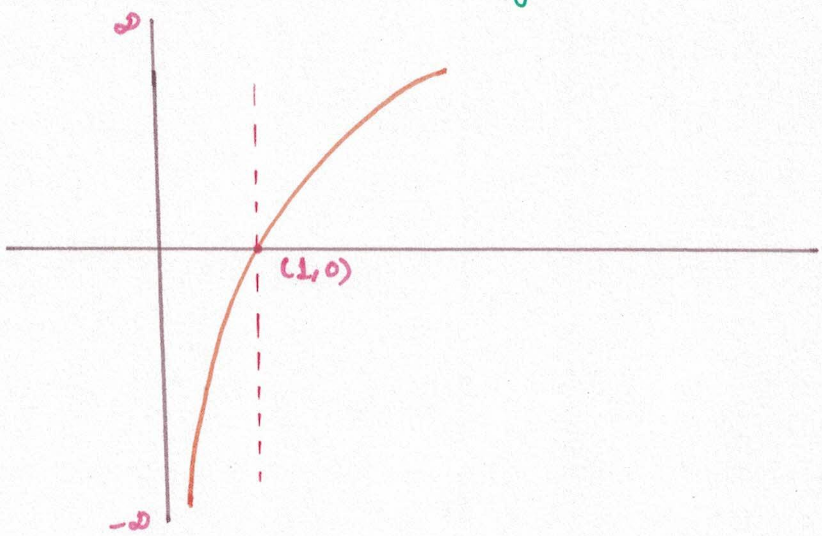


② $\frac{dy}{dx} = \cos x$ ③ $-\sin x = \frac{d^2y}{dx^2}$

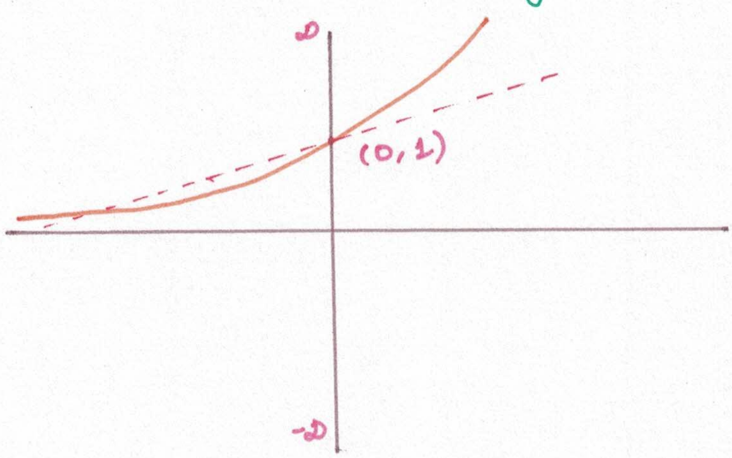
Question: $\ln x \quad x \in (0, \infty) \quad$ Range $(-\infty, \infty)$

$\frac{dy}{dx} = \frac{1}{x}$

$\frac{d^2y}{dx^2} = -\frac{1}{x^2}$



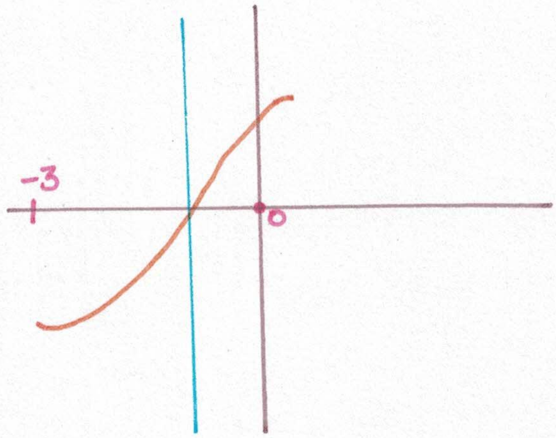
Question: $e^x \quad x \in R \quad$ Range $(0, \infty)$



Question: $y = \frac{x+1}{x^2+3}$

$x^2y - x + 3y - 1 = 0$

$12y^2 - 6y + 2y - 1 \le 0 \xrightarrow{D \ge 0} y \in \left[\frac{-1}{6}, \frac{1}{2}\right]$

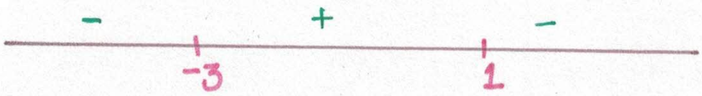


$$\frac{dy}{dx} = \frac{(x+1)(2x) - (x^2+3)}{(x^2+3)^2}$$

$$= \frac{2x^2+2x-x^2-3}{(x^2+3)^2}$$

$$= \frac{3-2x-x^2}{(x^2+3)^2}$$

$$= \frac{-(x+3)(x-1)}{(x^2+3)^2}$$

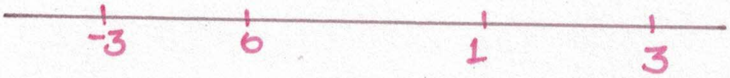


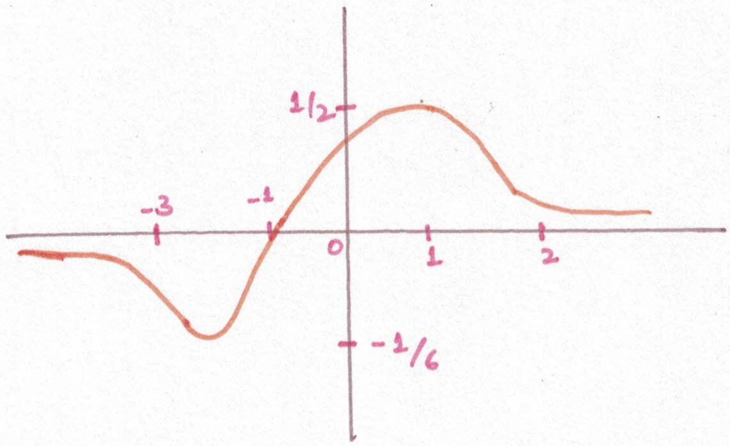
$$\frac{d^2y}{dx^2} = \frac{(x^2+3)^2(-2-2x) - (3-2x-x^2)(2x)(2)(x^2+3)}{(x^2+3)^3}$$

$$\frac{d^2y}{dx^2} = \frac{(x^2+3)(-2-2x) - (3-2x-x^2)(4x)}{(x^2+3)^3}$$

$$\frac{d^2y}{dx^2} = \frac{2x^3+6x^2-18x-6}{(x^2+3)^3}$$

$$\frac{d^2y}{dx^2} = \frac{2(x^3+3x^2-9x-3)}{(x^2+3)^3}$$





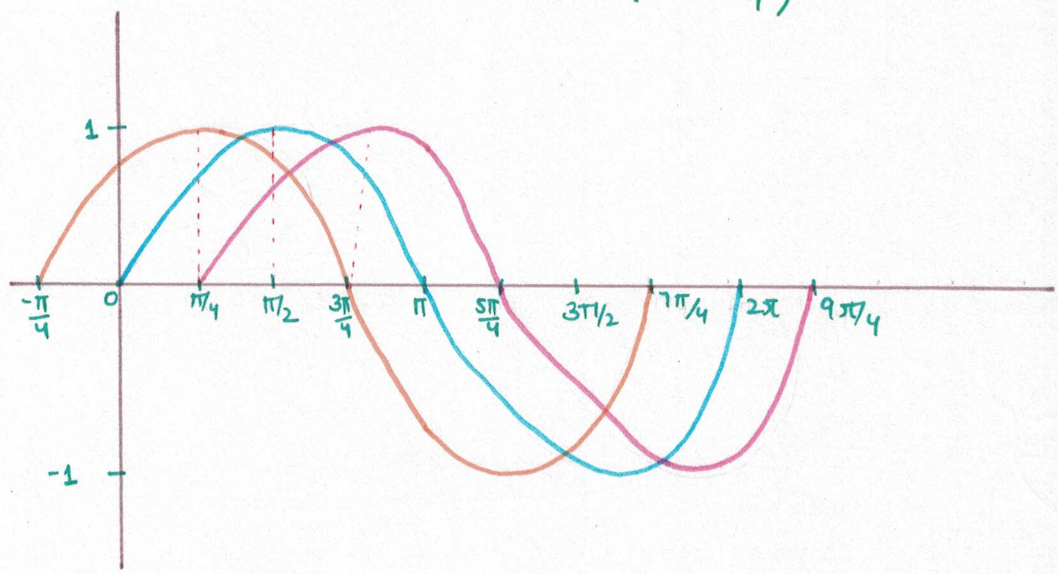
TRANSFORMATION OF GRAPHS

① $f(x \pm a)$, $a > 0$

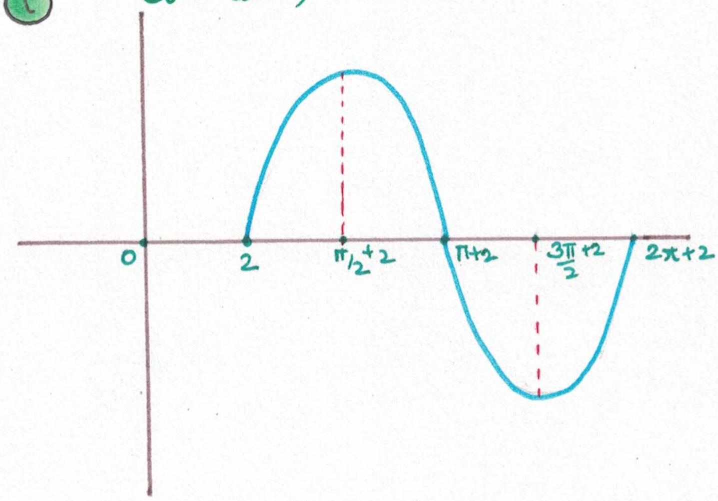
If $f(x)$ is a known graph, then we can draw the graph of $f(ax)$ by shifting the graph left or rightward by a units along x-axis.

If $+a$, then shifting towards left and if $-a$, then shift towards right.

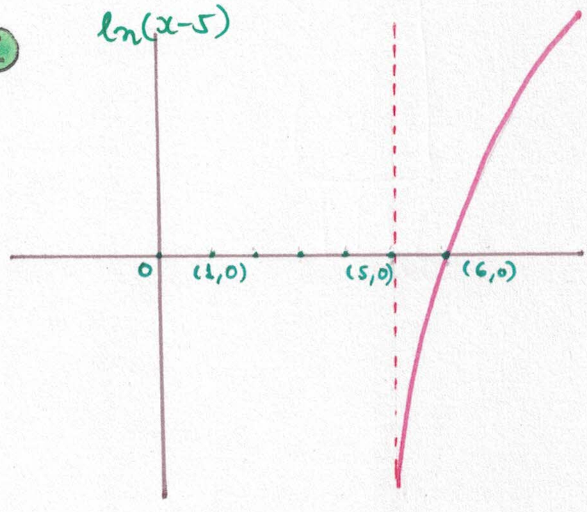
$$f(x) = \sin x$$
$$g(x) = \sin\left(x - \frac{\pi}{4}\right)$$
$$h(x) = \sin\left(x + \frac{\pi}{4}\right)$$



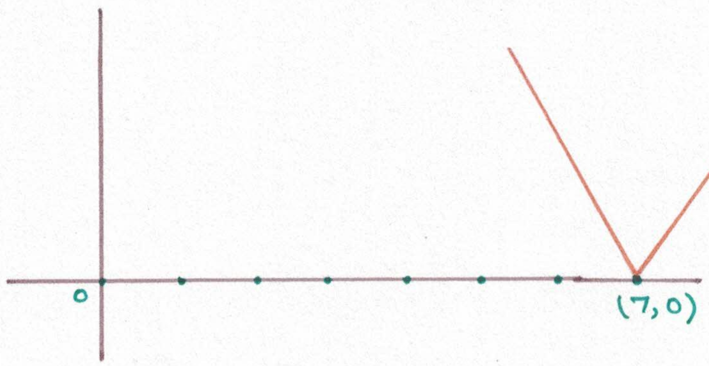
(i) $\sin(x-2)$



(ii) $\ln(x-5)$



(iii) $|x-7|$

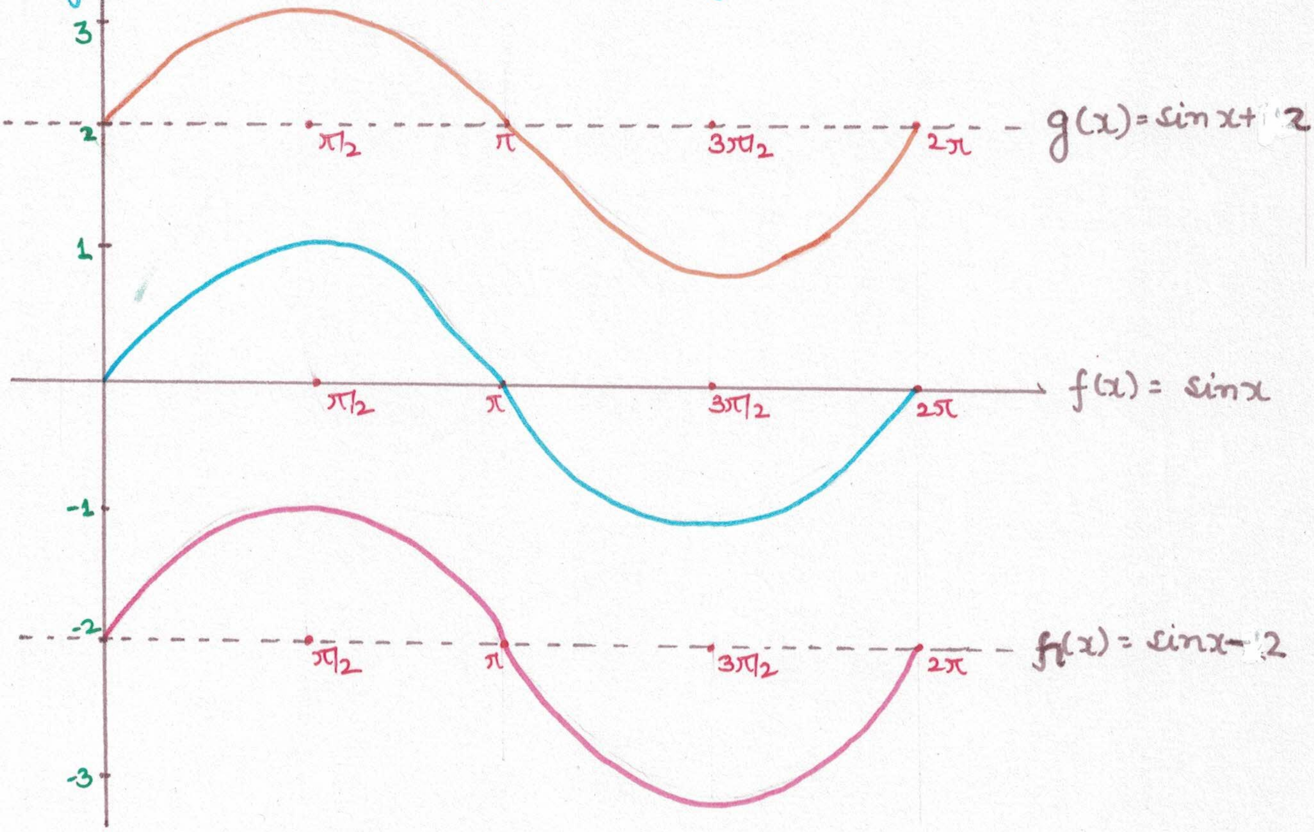


(2) $f(x) \pm a$

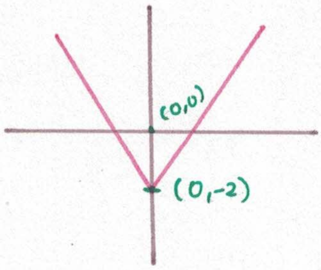
If $f(x)$ is a known graph, then we can draw the graph of $f(x) \pm a$ by shifting the graph upwards and downward by a unit along y axis.

If $-a$, then downward shift

If $+a$, then upward shift

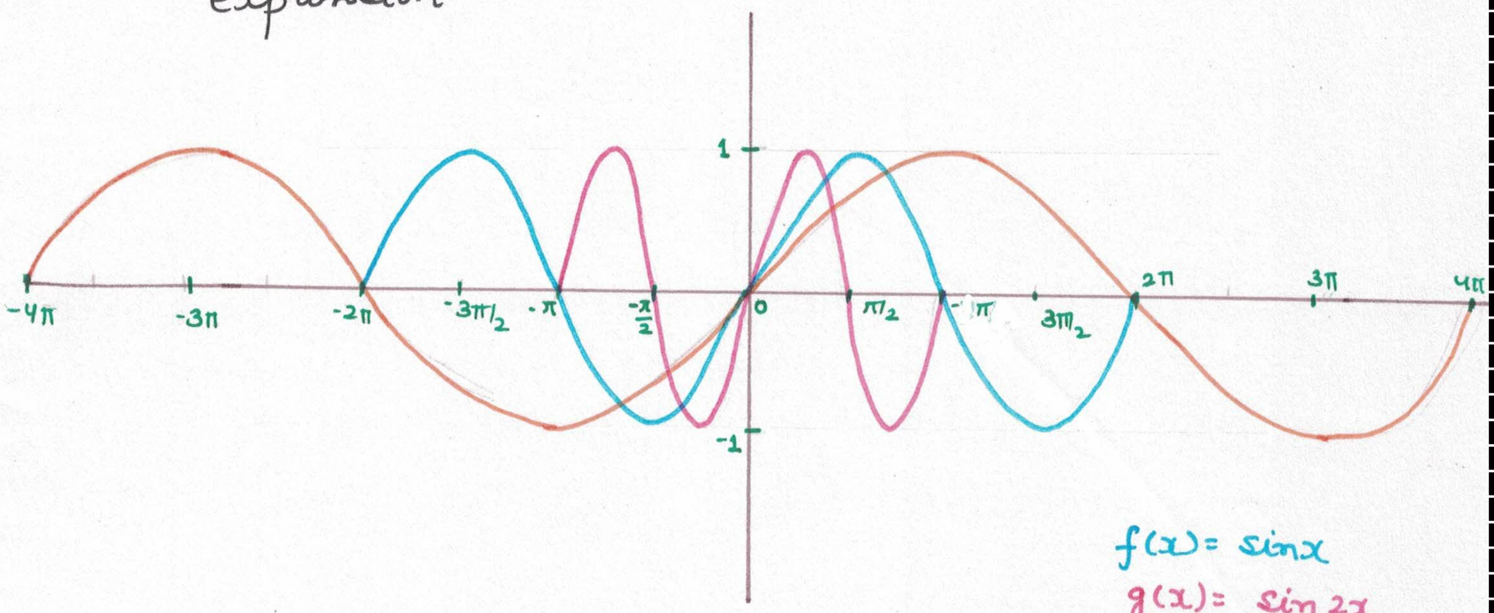


$|x| - 2$



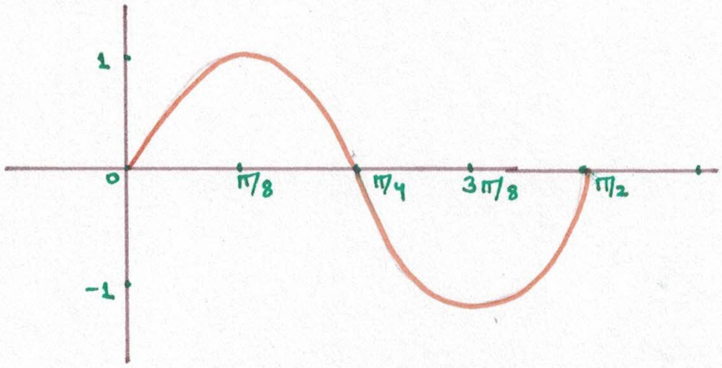
③ $f(ax)$ $a>0$

If $f(x)$ is a known graph to us, then we can draw the graph of $f(ax)$ by expanding or compressing the graph along x-axis by a units. If $a>1$, then compression and if $a<1$, then expansion.

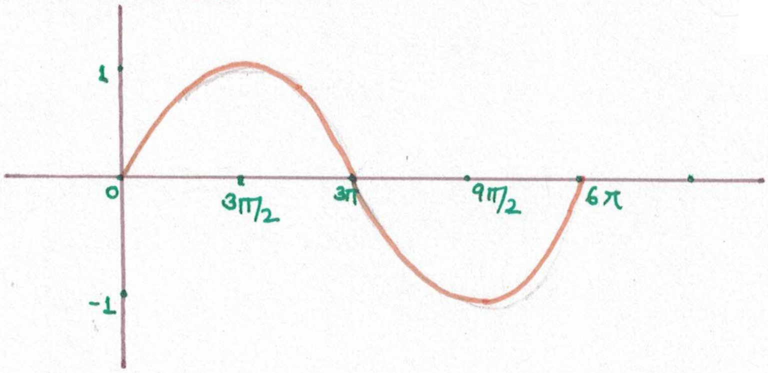


$f(x) = \sin x$
$g(x) = \sin 2x$
$h(x) = \sin \frac{x}{2}$

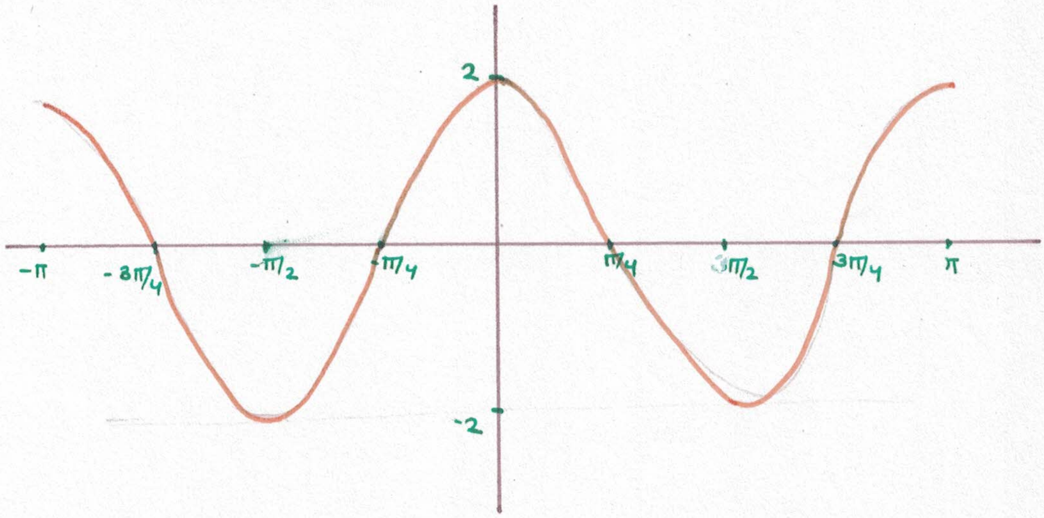
Ques: sin 4x



Ques: sin x/3



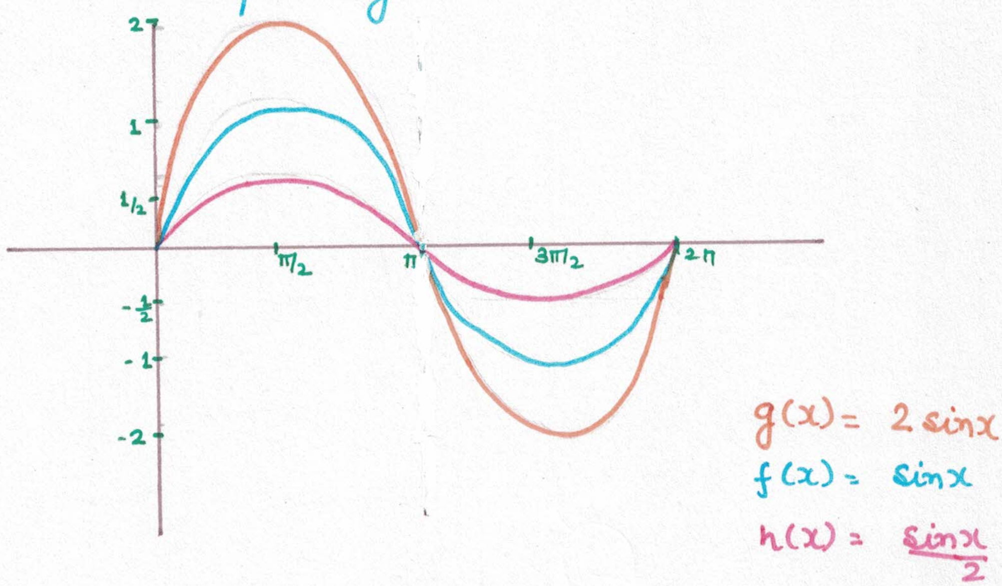
Ques: $2\cos^2 x = \cos 2x + 1$



4 $a f(x)$

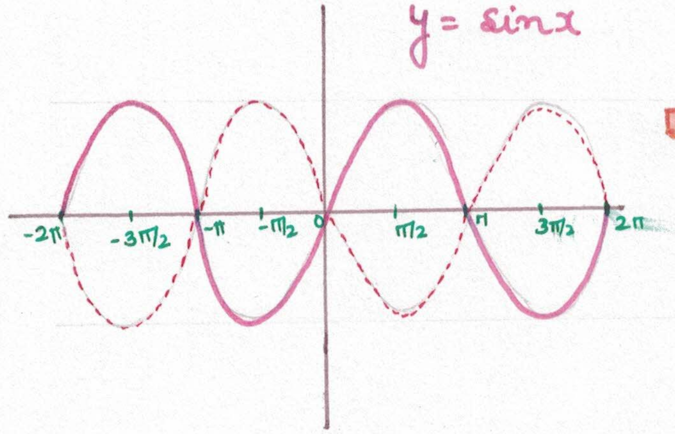
If $f(x)$ is a known graph, then we can draw graph of $af(x)$ by compressing and expanding the graph along y-axis by a units.

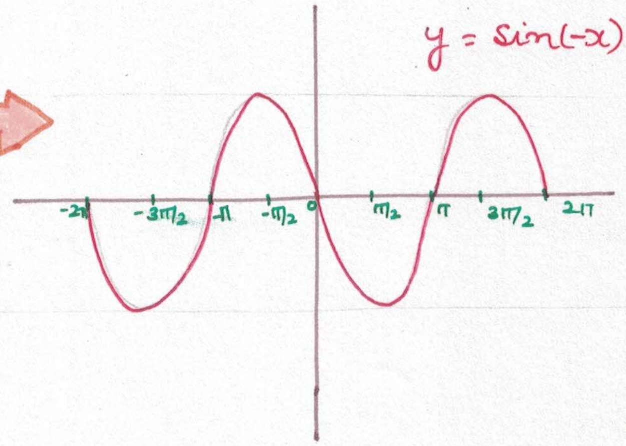
If $a > 1$, then expanding and
if $a < 1$, then compressing



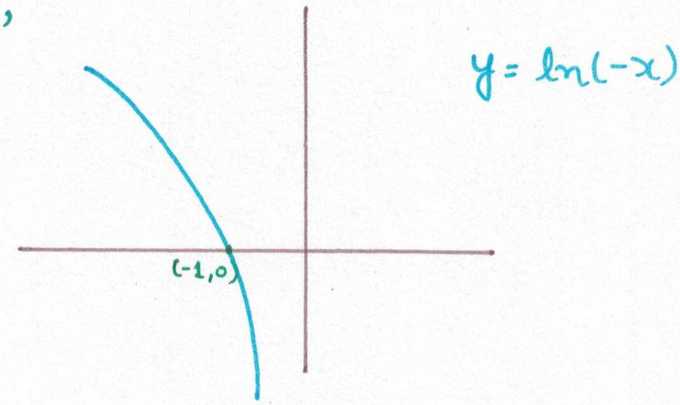
5 $f(-x)$

If $f(x)$ is a known graph, then we can draw the graph of $f(-x)$ by taking the image of $f(x)$ with respect to y-axis and then delete the graph of $f(x)$. The remaining graph is $f(-x)$



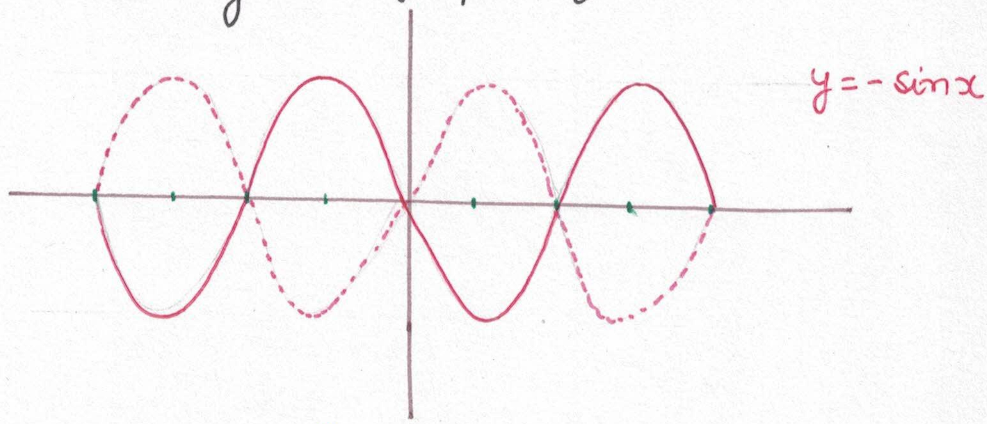


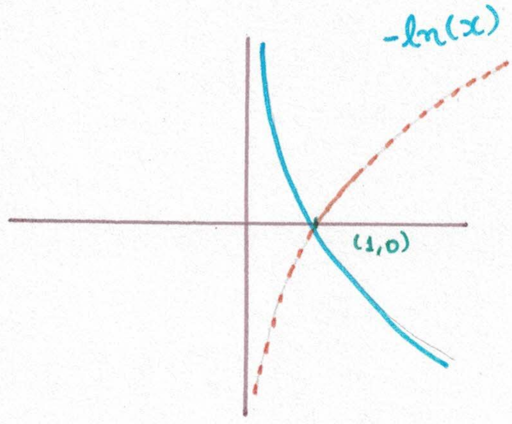
Similarly,

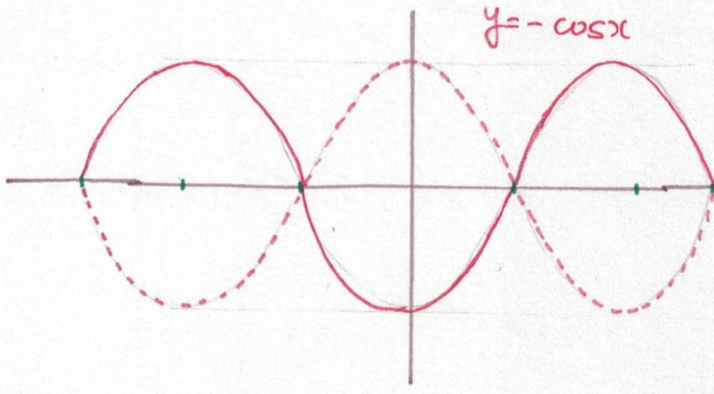


6 $-f(x)$

If $f(x)$ is a known graph then we can draw the graph of $-f(x)$ by taking the image of graph of $f(x)$ by x-axis and then deleting the graph of $f(x)$

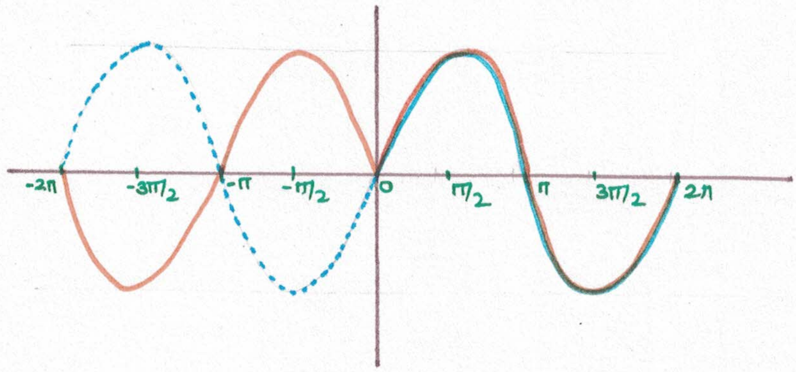




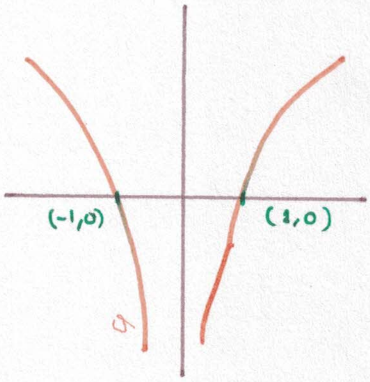


(7) $f(|x|)$

If $f(x)$ is a known graph, then we can draw the graph of $f|x|$ by deleting the graph of $f(x)$ which lies on -ve y axis, after that take the image of remaining graph of $f(x)$ with respect to y axis. The whole graph is known as $f(|x|)$



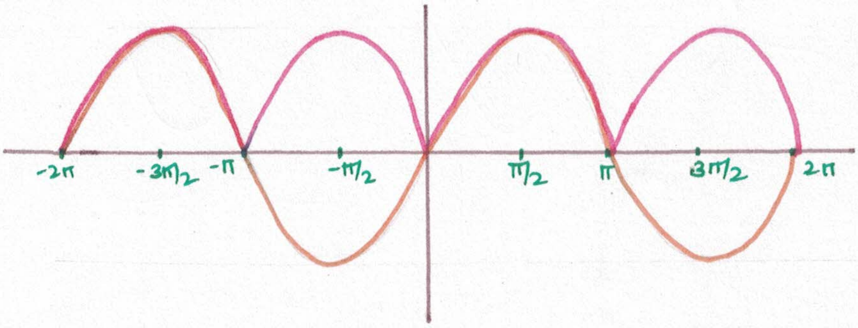
$f(x) = \sin x$
$g(x) = \sin|x|$



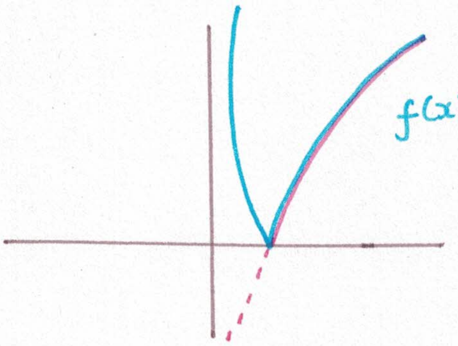
$\ln|x|$

(8) $|f(x)|$

If $f(x)$ is a known graph then we can draw the graph of $|f(x)|$ by taking the image of the graph which is lying on -ve y-axis (below x-axis) with the respect to x-axis and after that delete the graph of $f(x)$ which is lying below x-axis.

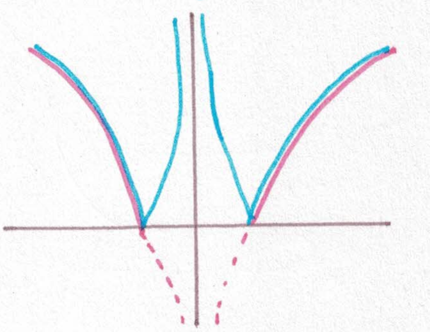


$f(x) = |\sin x|$



$f(x) = |\ln x|$

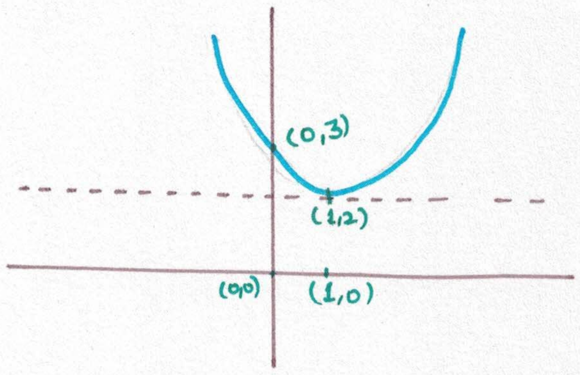
$f(x) = |\ln|x||$



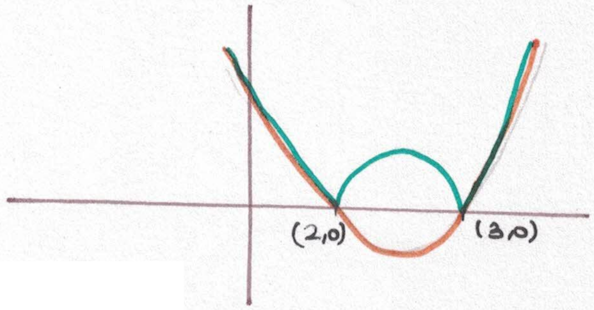
Ques: $f(x) = |3 - 2x + x^2|$

$x^2 - 2x + 3 = |(x-1)^2 + 2|$

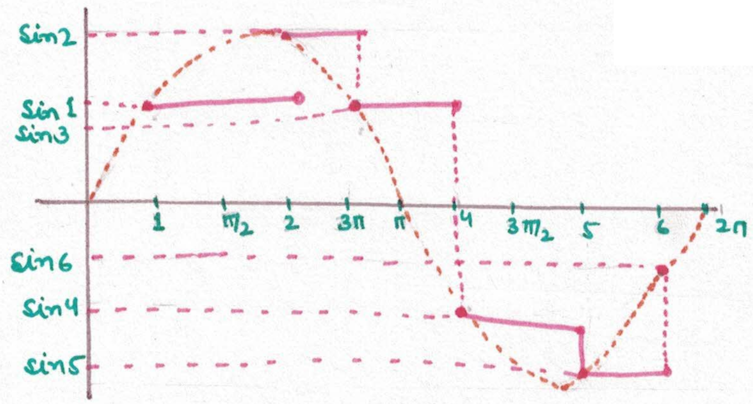
$y = (x-1)^2 = (1, 0)$



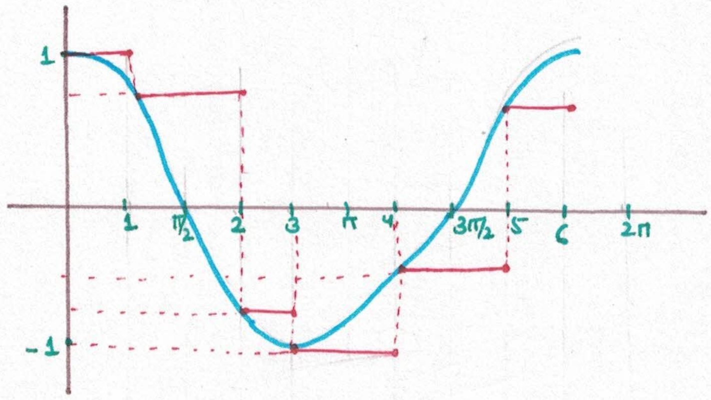
Ques: $y = |x^2 - 5x + 6|$



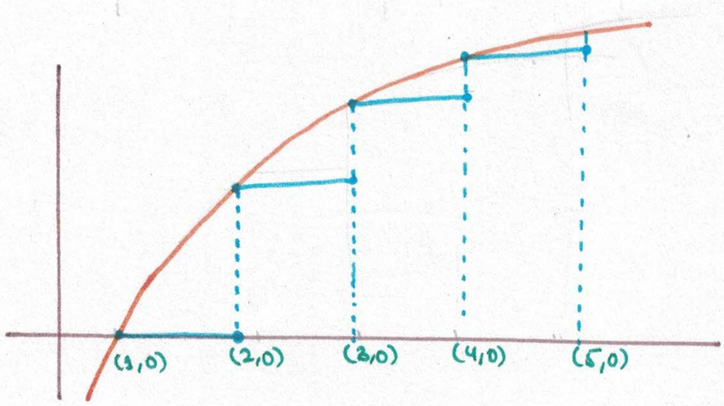
9 $y = f[x]$ $x \in [0, 2\pi]$



$y = \sin[x]$

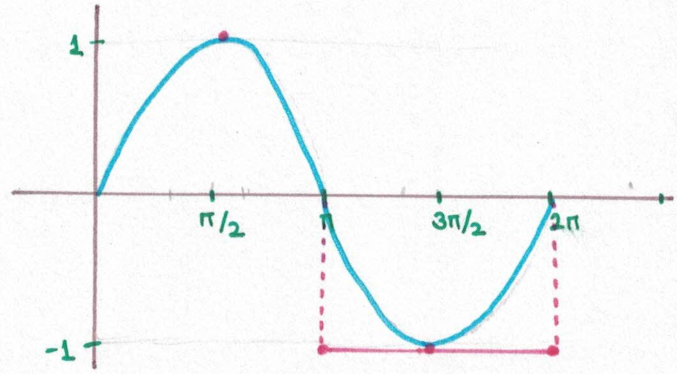


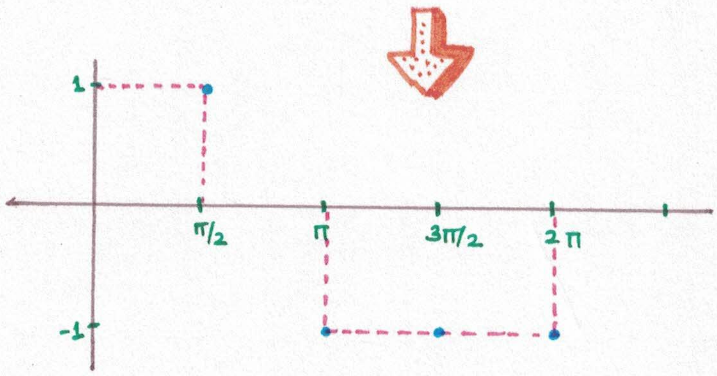
$y = \cos[x]$

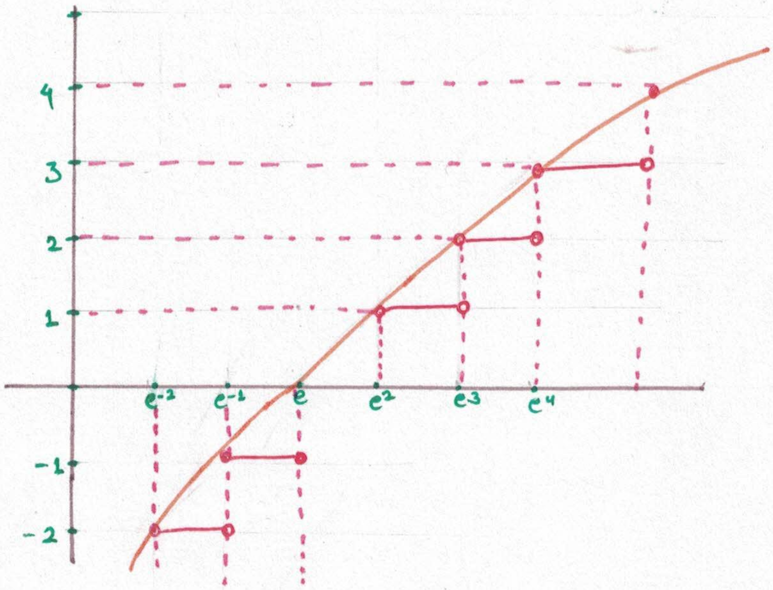


$y = \ln[x]$

10. $y = [f(x)]$







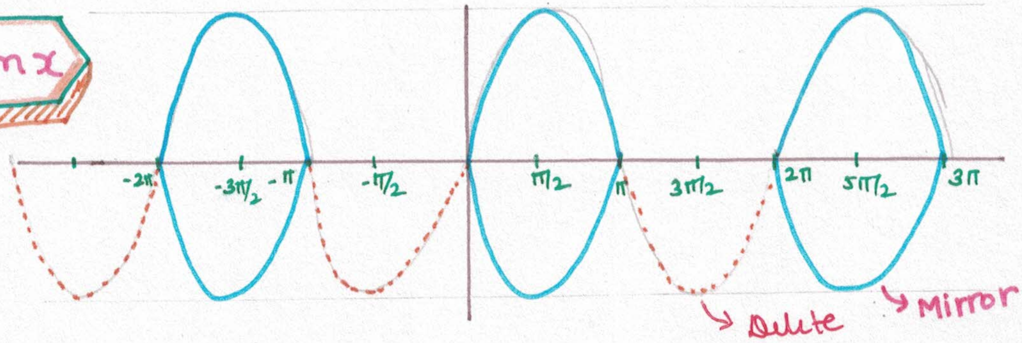
$y = [\ln x]$

11. $|y| = f(x)$

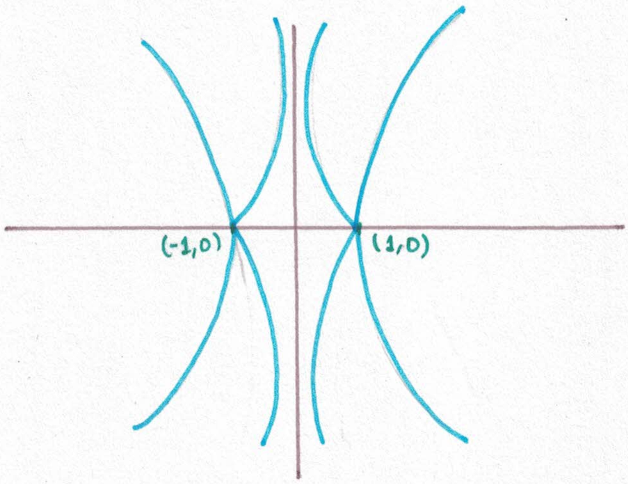
If $f(x)$ is a known graph to us, then we can draw the graph of $|y| = f(x)$ by deleting the graph of which lies below x-axis and then taking the image of remaining graph with respect to x-axis.

The collective graph is $|y| = f(x)$

$|y| = \sin x$



$|y| = |\ln|x||$

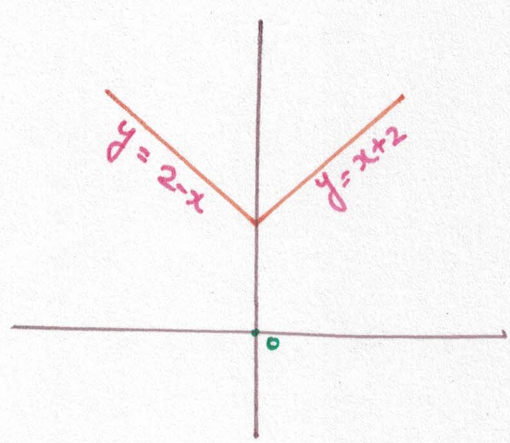


$$f(x) = \begin{cases} 2+x & x \geq 0 \\ 2-x & x < 0 \end{cases}, \quad \text{find } f(f(x))\,?$$

Sol: $$f(f(x)) = \begin{cases} 2+f(x) & f(x) \geq 0 \\ 2-f(x) & f(x) < 0 \end{cases}$$

$$f(f(x)) = \begin{cases} 2+2+x & x \geq 0 \\ 2+2-x & x < 0 \end{cases}$$

$$= \begin{cases} 4+x & x \geq 0 \\ 4-x & x < 0 \end{cases}$$

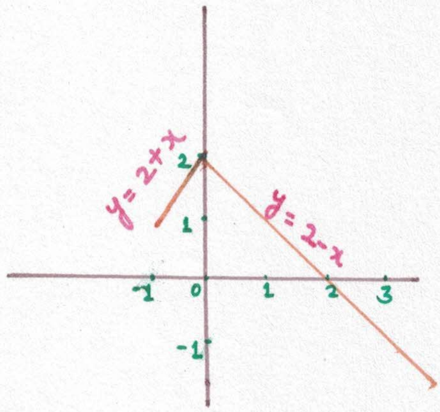


Ques: $f(x) = -1 + |x-2| \quad 0 \leq x \leq 4$

$g(x) = 2 - |x| \quad -1 \leq x \leq 3$, find $f(g(x))$?

Sol: $f(x) = \begin{cases} +1-x & 0 \le x < 2 \\ x-3 & 2 \le x \le 4 \end{cases}$

$g(x) = \begin{cases} 2+x & 4 \le x < 0 \\ 2-x & 0 \le x \le 3 \end{cases}$



$f(g(x)) = \begin{cases} 1-g(x) & 0 \le g(x) \le 2 \\ g(x)-3 & 2 \le g(x) \le 4 \end{cases}$

$f(g(x)) = \begin{cases} 1-(2+x) & -1 \le x \le 0 \\ 1-(2-x) & 0 < x \le 2 \end{cases}$